Clustered Flow Shop Models

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Abstract

This paper deals with flow-shop models where n items are grouped in fixed sequences, called clusters. The clusters are to be processed on m machines in the same technological order. Each machine handles the clusters in the same order. Each completed item is available for processing on the next machine. The question is how to arrange the clusters to minimize the completion time. Models that include setup times for clusters with identical items are also examined. Approximate solutions along with new lower bounds are presented. Those solutions are optimal for the two machine case.

1. Introduction

This paper considers a class of flow-shop sequencing models, referred to as the clustered flow-shop problems. Consider an example problem, the set of n items to be processed has been partitioned into k categories or clusters of identical items. Machine set up times are incurred whenever processing is switched from items of one cluster to items of the next cluster. The magnitude of such setup times is significant enough to warrant processing all items in a cluster before processing any items from another cluster. Once a sequence of items within a cluster is fixed the problem reduces to one of sequencing clusters rather than sequencing individual items. Clustered problems arise in practice in a variety of situations. Clusters may represent a set of items to be shipped to a different destination. The items of a particular cluster may be required parts for assembly of a specific module. The arrangement of items within the clusters would correspond to the assembly order.

This problem cannot be solved by existing flow-shop techniques. To solve this and other related problems that involve nonidentical items within a cluster we formulate three models.

Model I assumes that the n items are grouped in k fixed sequences or clus ters $\alpha_1, \alpha_2, \ldots, \alpha_k$. Those clusters are to be processed on m machines in the same

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technological order $M_1, M_2, \ldots, M_m$. Each completed item is ready for processing on the next machine. The objective is to arrange the clusters to minimize makespan. We also consider Model II where the order of items within some clusters is arbitrary.

Model III is a generalization of Model I which incorporates time lags before processing some or all of the items. This model covers clustered problems that involve setup times which can be viewed as negative time lags (see [11]).

Model I was first formulated by T. Kurisu [3] who solved the two machine case by treating it as a special case of Mitten's time lag model [8]. He also solved four special three machine cases [5]. In [4] Kurisu considered a two machine problem where the set of clusters is partially ordered. P. L. Maggu and G. Das [6] demonstrated that the two machine case can be solved by considering instead an equivalent flow shop problem where clusters α_i are replaced by items a_i. They developed explicit formulas for a_i wherever α_i is a two item cluster. Das [1] extended this method to two special three machine cases of [9]. Maggu, Das and Kumar [7] used the same approach to solve the two machine Model III with positive time lags (corresponding to transport times from one machine to the next).

This author examined in [12] a multimachine case of Model I and provided conditions when the clustered version can be reduced to a classical flow shop case. He also derived an approximate solution along with a lower bound.

A simple way to find the optimal solution of the clustered problem is to examine all k! sequences. This approach is quite costly, however, for problems with $k \geq 15$ even with modern computers.

This paper presents approximate solutions of Models I, II and III that are optimal for the two machine case. Each model is treated separately in Sections 2, 3, and 4. We derive new lower bounds based on the clustered structure that are stronger than that of [12]. The solution of the example problem which is a special case of Model III has been presented in Section 5. This section also considers properties of a related problem with zero setup times.

2. Approximate Solution and New Lower Bound of Model I

We adopt the following notations of [12]:

t_{rs} - the processing time of item r on M_s, $1 \le r \le n$, $1 \le s \le m$.

$T(\alpha,m)$ - the completion time of sequence α processed on all m machines.

We assume that processing of α starts at zero.

$C(\alpha,u,v)$ - the processing time of sequence α handled by $M_u, M_{u+1}, \ldots, M_v$.

Then $C(\alpha,1,m) = T(\alpha,m)$, $C(\alpha,u,u) = \sum_{r \in \alpha} t_{ru}$.

$C^s_{(r)}$ - the completion time of the r-th item of a given sequence processed on $M_1, M_2, \ldots, M_s$ (processing of this sequence starts at zero) and $t_{(r),s}$ - the processing time of this item on M_s.

It is well known [2] that

$$C^s_{(r)} = \max[C^{s-1}_{(r)}, C^s_{(r-1)}] + t_{(r),s}, \quad C^0_{(r)} = C^s_{(0)} = 0, \quad \forall \; 1 \le r \le n, \; 1 \le s \le m. \tag{1}$$

Let Γ be a m+n-1 sequence of cells (r,s) of a n × m matrix $\{t_{rs}\}$. We say that Γ is a <u>path</u> if it originates at (1,1), ends at (n,m) and makes steps to the right or downward. Assume for convenience that $P = 1,2,\ldots,n = \alpha_1, \alpha_2, \ldots, \alpha_k$ where $\alpha_i = p_i, p_i+1, \ldots, q_i$. According to [12],

$$T(P,m) = \max_{\Gamma \in [\Gamma]} \sum_{(r,s) \in \Gamma} t_{rs}, \tag{2}$$

where $[\Gamma]$ is a set of all possible paths, and $\sum_{(r,s) \in \Gamma} t_{rs}$ is the length of path Γ.

To find T(P,m) for the clustered model, it is convenient to consider a path Γ that: a) enters the α_i area (which occupies rows $p_i, p_i+1, \ldots, q_i$) in column w_{i-1}, $2 \le i \le k$, b) consists of segments $\gamma_1, \gamma_2, \ldots, \gamma_k$ whose lengths are $C(\alpha_1,1,w_1)$, $C(\alpha_2,w_1,w_2)$ and $C(\alpha_k,w_{k-1},m)$ respectively.

It is easy to see that

$$T(P,m) = \max_{1 \le w_1 \le w_2 \le \ldots \le w_{k-1} \le m} [C(\alpha_1,1,w_1) + C(\alpha_2,w_1,w_2) + \ldots + C(\alpha_k,w_{k-1},m)]. \tag{3}$$

Assume that the w_i of (3) are equal to 1 for $i \le v$ and m for $i > v$. Then

$$\left.\begin{aligned} T(P,m) &\ge \max_{1 \le v \le k} [\sum_{i=1}^{v-1} C(\alpha_i,1,1) + T(\alpha_v,m) + \sum_{i=v+1}^{k} C(\alpha_i,m,m)] = \\ &= \max_{1 \le v \le k} (\sum_{i=1}^{v} A_i - \sum_{i=1}^{v-1} B_i) + \sum_{i=1}^{k} \sum_{r \in \alpha_i} t_{rm} \end{aligned}\right\} \tag{4}$$

where

$$A_i = T(\alpha_i,m) - \sum_{r \in \alpha_i} t_{rm}, \quad B_i = T(\alpha_i,m) - \sum_{r \in \alpha_i} t_{r1}, \quad 1 \le i \le k. \tag{5}$$

For m=2, formula (4) is an equality since all w_i are 1 or 2.

Let L(P) be the right hand side of (4).

To arrange the clusters in a sequence that minimizes L(P), apply

<u>Procedure I</u>: Use Johnson's Algorithm [2] to solve a two-machine flow-shop problem (A_i, B_i) where A_i and B_i, defined by (5), are processing times of "item" i on the first and second machine.

The resulting sequence, say P, is the approximate solution of Model I and the lower completion time bound $LB_1 = \min_P L(P)$. Notice that $T(P,m) = LB_1$ for m=2.

P is optimal (i.e., $T(P,m) = LB_1$) whenever one of its critical paths moves to the right only in rows that correspond to a single cluster.

One can show that the derivation of the lower bound of the clustered model based on the formula T(P,m) of [12], page 320 (rather than (3)) leads to a lower bound, $LB_1' = \min_P L(P)$ with A_i and B_i defined by

$$A_i = D(\alpha_i) - \sum_{r\in\alpha_i} t_{rm}, \quad B_i = D(\alpha_i) - \sum_{r\in\alpha_i} t_{r1}, \qquad 1\le i\le k, \tag{5'}$$

where

$$D(\alpha_i) = \max_{p_i\le u\le q_i} \left(\sum_{r=p_i}^{u} t_{r1} + \sum_{s=2}^{m-1} t_{us} + \sum_{r=u}^{q_i} t_{rm} \right).$$

Due to $T(\alpha_i, m) \ge D(\alpha_i)$ for each i, $LB_1 \ge LB_1'$.

Also notice that the optimality condition $T(P,m) = LB_1'$ is much more <u>stringent</u> than $T(P,m) = LB_1$ since it requires that one of the critical paths of P pass an <u>entire</u> row. Hence LB_1 is a considerably better bound than LB_1'.

Procedure I is illustrated by the following example:

<u>Example 1</u>: Consider a four machine problem where $\alpha_1 = (1,2,3)$, $\alpha_2 = (4,5)$, $\alpha_3 = (6,7)$, $\alpha_4 = (8,9,10)$. The t_{rs} are given in Figure 1.

r \ s	M_1	M_2	M_3	M_4
1	12	13	8	4
2	11	7	6	5
3	7	5	4	8
4	4	3	11	9
5	7	8	12	7
6	5	6	10	6
7	11	7	9	7
8	10	15	10	6
9	11	12	5	7
10	7	4	15	12

Figure 1.

Apply (1) to find $T(\alpha_i,m)$ for i=1,2,3,4. As a result we get $T(\alpha_1,m) = 52$, $T(\alpha_2,m) = 38$, $T(\alpha_3,m) = 39$, and $T(\alpha_4,m) = 69$. The A_i and B_i defined by (5) are given in Figure 2a.

α_i	A_i	B_i
α_1	35	22
α_2	22	27
α_3	26	23
α_4	44	41

Figure 2a

α_i	A_i	B_i
α_2	22	27
α_4	44	41
α_3	26	23
α_1	35	22

Figure 2b

Procedure I produces sequence $P = \alpha_2\alpha_4\alpha_3\alpha_1$. Using (4), we can see from Figure 2b that $L(P) = \max(22,66-27,92-68,127-91)+(17+16+13+25) = 110$.

Utilizing (1) for P = 4,5,8,9,10,6,7,1,2,3, find T(P,4) = 114. The respective critical path Γ = (1,1),(2,1),(3,1),(3,2),(4,2),(4,3),(5,3),(6,3),(7,3),(8,3),(9,3),(9,4),(10,4) moves to the right in rows 3, 4, and 9 occupied by clusters α_4 and α_1. This is why $LB_1 < T(P,m)$. The optimal solution of this example is sequence $P'=\alpha_2\alpha_3\alpha_4\alpha_1$ where T(P',4) = 113. Applying Procedure 1 with A_i and B_i defined by (5'), we get $LB_1' = 104$.

Remark: If $LB_1 < T(P,m)$ and the critical path of P makes its first and last moves to the right in rows r_1 and r_2 (where r_1 is considerably smaller than r_2) occupied by different clusters we suggest to also consider the known machine based bound. For the clustered case this bound becomes

$$LB_1^o = \max_{1\le u\le m} \left[\sum_{r=1}^{n} t_{ru} + \min_{1\le i\ne j\le k} \left(\sum_{s=1}^{u-1} t_{p_i s} + \sum_{s=u+1}^{m} t_{q_j s} \right) \right].$$

In the ten item Example 1, $LB_1^o = 105$.

Notice that for each special case of [1] and [5] the critical path of every permutation passes an entire row or an entire column. Then the completion time of the resulting solution is either LB_1' or LB_1^o.

3. The Case When Some Clusters are Arbitrary Sequences (Model II)

Assume that the order of items is arbitrary for certain clusters α_i, where $i\in I \subset (1,2,\ldots,k)$. Hence the question is how to arrange the clusters as well as the items within each cluster α_i, $i\in I$, in order to minimize the completion time.

To find an approximate solution and the lower completion time bound LB_2 apply

Procedure II:

Step 1. For each $i\in I$ use Johnson's Algorithm to find sequence (cluster) α_i - the optimal solution of a two machine problem

(t_{r1}', t_{r2}'), $p_i\le r\le q_i$, where $t_{r1}' = \sum_{s=1}^{m-1} t_{rs}$, $t_{r2}' = \sum_{s=2}^{m} t_{rs}$. Let $T'(\alpha_i)$ be the completion time of α_i for this two machine problem. According to [9],

$T'(\alpha_i) - \sum_{r\in\alpha_i} \sum_{s=2}^{m-1} t_{rs}$ is the lower completion bound $LB(\alpha_i)$ of cluster α_i.

Step 2. Once all clusters are <u>fixed</u> sequences, apply Procedure I to arrange the α_i using the A_i and B_i, given by (5) for $i \notin I$, and

$A_i = LB(\alpha_i) - \sum_{r \in \alpha_i} t_{rm}$, $B_i = LB(\alpha_i) - \sum_{r \in \alpha_i} t_{r1}$, for $i \in I$, where $LB(\alpha_i)$ is defined in Step 1.

Let $L'(P)$ be the right hand side of (4) where the $T(\alpha_i, m)$ are replaced by $LB(\alpha_i)$ whenever $i \in I$.

It is obvious that $L'(P) \le L(P) \le T(P,m)$ for every P. If P is the sequence produced by Procedure II then $LB_2 = \min_P L'(P)$.

Consider the two machine case. It is easy to show that Procedure II provides an optimal solution of the clustered problem.

If $T(P,m) > LB_2$ the use of better bounds $LB(\alpha_i)$ in Step 2 of Procedure II might result in an increase of LB_2. We illustrate this procedure of Example 1 assuming that set I = (1,4). According to Step 1, $\alpha_1 = 3,1,2$, $\alpha_4 = 10,8,9$, $T'(\alpha_1) = 92$, and $T'(\alpha_4) = 116$. Hence $LB(\alpha_1) = 49$, and $LB(\alpha_4) = 55$. Execution of Step 2 results in $P = \alpha_2\alpha_4\alpha_3\alpha_1 = 4,5,10,8,9,6,7,3,1,2$ where $L'(P) = LB_2 = 104$, and $T(P,m) = 106$. Notice that the machine based bound for this case is 101. Applying Procedure I with A_i and B_i defined by (5) we get the same sequence $P = \alpha_2\alpha_4\alpha_3\alpha_1$ and $L(P) = 106 = T(P,m)$. Hence P is an optimal solution. Notice that critical path of P makes all its right turns in the last two rows of cluster α_1.

4. <u>The Clustered Model with Time Lags (Model III)</u>

This author considered in [11] a generalization of the flow-shop model by replacing (1) with

$$C^s_{(r)} = \max[C^{s-1}_{(r)} + a_{(r),s}, C^s_{(r-1)}] + t_{(r),s}, \quad \forall \ 1 \le r \le n, \ 1 \le s \le m. \qquad (1')$$

where $C^0_{(r)} = C^s_{(0)} = 0$ and $a_{(r),s}$, called time lag, is an arbitrary real number with $a_{(r),1}=0$ for each r.

Let P = 1,2,...,n. Also assume

$$t_{rs} + a_{r,s+1} \ge 0, \quad \forall \ 1 \le r \le n, \ 1 \le s \le m. \qquad (6)$$

This condition automatically holds for nonnegative a_{rs}.

Let $\Gamma(0) = [(u_1,2),(u_2,3),\ldots,(u_{m-1},m)]$, $1 \le u_1 \le u_2 \le \ldots \le u_{m-1} \le n$, be a set of cells where path Γ makes a step to the right. According to [11],

$$T(P,m) = \max_{\Gamma \in [\Gamma]} \left(\sum_{(r,s) \in \Gamma} t_{rs} + \sum_{(r,s) \in \Gamma(0)} a_{rs} \right). \qquad (7)$$

Let Model III be a clustered version of model (1'). It is easy to show that $T(P,m)$ is defined by (3) where the $C(\alpha_i, w_{i-1}, w_i)$ are calculated from (1'). Hence one can use Procedure I to find the approximate solution and the lower completion time bound. If (6) does not hold then

$$T(P,m) = \max_{1\le j\le m} \max_{\Gamma_j \epsilon [\Gamma_j]} \left(\sum_{(r,s)\epsilon\Gamma_j} t_{rs} + \sum_{(r,s)\epsilon\Gamma_j(0)} a_{rs} \right),$$

where Γ_j is a path that originates at cell (1,j) and ends at (n,m), while $\Gamma_j(0)$ is a set of cells where Γ_j turns to the right. For the clustered case formula (7) leads to

$$T(P,m) = \max_{1\le j\le m} \max_{j\le w_1\le w_2\le \ldots \le w_{k-1}\le m} [C(\alpha_1, j, w_1) + C(\alpha_2, w_1, w_2) + \ldots + C(\alpha_k, w_{k-1}, m)].$$

One can show (as in Section 2) that

$$T(P,m) \ge \max_{1\le j\le m-1} [\max_{1\le v\le k} (\sum_{i=1}^{v} A_{ij} - \sum_{i=1}^{v-1} B_{ij}) + \sum_{i=1}^{k} \sum_{r\epsilon\alpha_i} t_{rm}] \overset{df}{=} \max_{1\le j\le m-1} L_j(P), \qquad (4')$$

$$\text{where } A_{ij} = C(\alpha_i, j, m) - \sum_{r\epsilon\alpha_i} t_{rm}, \quad B_{ij} = C(\alpha_i, j, m) - \sum_{r\epsilon\alpha_i} t_{rj}. \qquad (5")$$

For m=2, inequality (4') becomes an equation. To find the lower completion time bound, use

<u>Procedure III</u>: Apply Johnson's Algorithm to solve m-1 two machine flow-shop problems (A_{ij}, B_{ij}), $1\le j\le m-1$, with A_{ij} and B_{ij} defined by (5"). Let L_j be the minimal value of $L_j(P)$. Since $T(P,m) \ge \sum_{r=1}^{n} t_{rm}$, the lower completion time bound is $\max(L_1, L_2, \ldots, L_{m-1}, \sum_{r=1}^{n} t_{rm})$. As an approximate solution choose sequence P_j with the smallest $T(P_j,m)$ among the m-1 generated sequences.

Introduce the following notations:

d_{rs}, p_{rs} - the setup and processing times of item r on M_s,

b_{rs} - the transportation time of moving item r from M_{s-1} to M_s for processing.

According to [11], model (1') covers flow-shop problems that involve:

a) the setup times d_{rs} by setting

$$t_{rs} = p_{rs} + d_{rs}, \quad a_{rs} = -d_{rs}, \quad s>1, \text{ and } a_{r1}=0, \qquad (8)$$

b) the transportation times b_{rs} by setting

$$t_{rs} = p_{rs}, \quad a_{rs} = b_{rs}, \quad s>1, \text{ and } a_{r1}=0. \qquad (9)$$

c) the setup and transportation times by setting

$$t_{rs} = p_{rs} + d_{rs},\ a_{rs} = b_{rs} - d_{rs},\ s>1,\ a_{r1}=0. \tag{10}$$

Model III can be handled by Procedures I or III depending on whether or not (6) is satisfied.

Example 2: Consider a clustered model with setup times where the d_{rs} and p_{rs} of one of the clusters, say, $\alpha_1 = (1,2,3)$ are given in Figure 3a.

r \ s	M_1	M_2	M_3	M_4
1	2,3	7,8	6,5	1,3
2	3,1	9,3	3,4	2,4
3	4,2	8,6	4,2	3,6

Figure 3a

r \ s	M_1	M_2	M_3	M_4
1	0,5	-7,15	-6,11	-1,4
2	0,4	-9,12	-3,7	-2,6
3	0,6	-8,14	-4,6	-3,9

Figure 3b

To find A_{11} and B_{11} of (5"), calculate first the a_{rs} and t_{rs} (see Figure 3b). Notice that condition (6) is violated since $t_{11} + a_{12} = 5-7 < 0$. Applying (1'), find $C(\alpha_1,1,4) = 44$. From (5") we get $A_{11} = 44 - (4+6+9) = 25$, $B_{11} = 44 - (5+4+6) = 29$.

5. Models with Identical Items in Each Cluster.

Consider the model with identical items in each cluster, stated in the Introduction. Let $d_{\alpha_i s}$ be the setup time on M_s to process the items of α_i. Also let $p_{\alpha_i s}$ be the processing time of every item of α_i on M_s.

This problem is a special case of Model III. To define the d_{rs} and p_{rs}, examine an arbitrary item $r \epsilon \alpha_i$. Set

$$d_{rs} = \begin{cases} d_{\alpha_i s}, & \text{if } r \text{ is the first item of } \alpha_i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_{rs} = p_{\alpha_i s},\quad \forall\ r \epsilon \alpha_i ,$$

The t_{rs} and a_{rs} are specified by (8).

We illustrate the solution of this model by the following:

<u>Example 3</u>: Let k=4, s=3, and the number of items of clusters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is 3, 4, 3, and 2 respectively.

Label $\alpha_1=(1,2,3)$, $\alpha_2=(4,5,6,7)$, $\alpha_3=(8,9,10)$, and $\alpha_4=(11,12)$.

The $d_{\alpha_i s}$ and $p_{\alpha_i s}$ are given in Figures 4a and 4b.

i \ s	M_1	M_2	M_3
α_1	2	8	7
α_2	5	3	4
α_3	10	9	6
α_4	4	7	9

Figure 4a

i \ s	M_1	M_2	M_3
α_1	5	9	6
α_2	9	11	12
α_3	7	18	11
α_4	3	4	6

Figure 4b

First find the t_{rs} and a_{rs} which are given for α_1 in Figures 5a and 5b.

	M_1	M_2	M_3
1	7	17	13
2	2	8	7
3	2	8	7

Figure 5a

	M_1	M_2	M_3
1	0	-9	-6
2	0	0	0
3	0	0	0

Figure 5b

As (6) is violated ($t_{11}+a_{12} = 7-9<0$), apply Procedure III. Calculating the $C(\alpha_i,j,3)$ we get: $C(\alpha_1,1,3) = 38$, $C(\alpha_2,1,3) = 36$, $C(\alpha_3,1,3) = 52$, $C(\alpha_4,1,3) = 32$, $C(\alpha_1,2,3) = 40$, $C(\alpha_2,2,3) = 30$, $C(\alpha_3,2,3) = 51$, $C(\alpha_4,2,3) = 29$. According to (5'), $(A_{11},A_{21},A_{31},A_{41}) = (13,8,23,8)$, $(B_{11},B_{21},B_{31},B_{41}) = (21,29,15,7)$, $(A_{12},A_{22},A_{32},A_{42}) = (13,2,22,5)$, $(B_{12},B_{22},B_{32},B_{42}) = (7,7,6,11)$.

Procedure III produces sequences $P_1 = \alpha_4,\alpha_1,\alpha_3,\alpha_2$ and $P_2 = \alpha_2,\alpha_4,\alpha_1,\alpha_3$.

Since $\sum_{r=1}^{12} t_{r3} = 108$, $L_1 = 8+108=116$, $L_2 = 17+108=125$, the lower completion time bound is 125. Next find $T(P_1) = 133$ and $T(P_2) = 134$. Hence P_1 is the approximate solution.

Assume that all $d_{rs}=0$. Due to (8), $t_{rs}=p_{\alpha_i s}$ for each $r\epsilon\alpha_i$. Notice that

$$T(\alpha_i,m) = \sum_{s=1}^{j-1} t_{rs} + \sum_{r\epsilon\alpha_i} t_{rj} + \sum_{s=j+1}^{m} t_{rs}, \quad 1\le i\le k, \tag{11}$$

where $t_{rj} = \max_{1\le s\le m} t_{rs}$. One may wonder whether this model is equivalent to some classical flow-shop model with properly chosen processing times.

For m=2 we will prove

Property 1: The clustered model is equivalent to a k item flow-shop problem where $p_{\alpha_i 1}$, $p_{\alpha_i 2}$ are processing times of "item" α_i on the first and second machine.

Proof: Consider cluster α_i. If $t_{r1}\le t_{r2}$ for each $r\epsilon\alpha_i$ then according to (11) and (5), $A_i\le B_i$ $t_{r1}=p_{\alpha_i 1}=A_i$. If $t_{r1}\ge t_{r2}$, $r\epsilon\alpha_i$ then $A_i\ge B_i$ and $t_{r2}=p_{\alpha_i 2}=B_i$. Hence Procedure I produces the same sequence as Johnson's algorithm applied to the two machine problem $(p_{\alpha_i 1}, p_{\alpha_i 2})$.

It is interesting to note that Property 1 does not depend on the size of clusters α_i. One can show that this property cannot be extended to m>2 even if the t_{rs} are replaced by $\sum_{r\epsilon\alpha_i} t_{rs}$.

6. Final Remarks

Consider a classical n-item multimachine flow-shop problem. Apply Procedure I (of Section 2) by setting $\alpha_i=i$ for each $1\le i\le k=n$. Let P = 1,2,...,n be the resulting sequence and

$$T(P,m) = \sum_{i=1}^{p-1} t_{r1} + C(\alpha,1,m) + \sum_{i=q+1}^{n} t_{rm},$$

for some sequence α = p,p+1,...,q. Notice that cells (p,1) and (q,m) are on the critical path Γ (see (2)). Based on the finding of Section 2, we conclude that P is the optimal solution of the flow-shop model provided items p,p+1,...,q are processed in the indicated order as a cluster. If p=q then P is unconditionally optimal.

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